MATH2101 Complex Analysis (Year 2009/10)
Examination questions and solutions

The following notation is used throughout: for any $a \in \mathbb{C}$ and $r > 0$, $S(a,r)$ denotes a positively oriented circular contour of radius r, centred at a; $D(a,r) = \{z : |z-a| < r\}$, $\overline{D}(a,r) = \{z : |z-a| \le r\}$, $D'(a,r) = \{z : 0 < |z-a| < r\}$.

1. (a) Suppose that F is an entire function, and that $F(z) = F(-z)$ for all $z \in \mathbb{C}$. Using Taylor's series for F at $z_0 = 0$, show that there exists an entire function G such that $F(z) = G(z^2)$.

   (b) Let f be an analytic function on the domain $D'(0,2)$, and suppose that for all $n = 0, 1, 2, \ldots$,
   $$\int_{S(0,1)} z^n f(z)\,dz = 0.$$
   Show that f has a removable singularity at $z = 0$.

   (c) Find the Laurent expansions for the function
   $$f(z) = \frac{z}{(z-2i)(z+1)}$$
   valid for (i) $0 < |z+1| < \sqrt{5}$, (ii) $|z| > 2$.

Solution.

(a) Write Taylor's series for F:
$$F(z) = \sum_{k=0}^{\infty} a_k z^k,$$
and split it in two parts: $F(z) = F_1(z) + F_2(z)$, with
$$F_1(z) = \sum_{j=0}^{\infty} a_{2j} z^{2j}, \quad F_2(z) = \sum_{j=0}^{\infty} a_{2j+1} z^{2j+1}.$$

Clearly, $F_1(z) = F_1(-z)$ and $F_2(z) = -F_2(-z)$. In view of the condition $F(z) = F(-z)$, we have $F_2(z) = 0$. Thus, with the entire function $G(z) = \sum_{j=0}^{\infty} a_{2j} z^j$ we have $F(z) = G(z^2)$, as required.

(b) By Laurent Theorem, f can be expanded in a Laurent series absolutely converging on the domain D:

$$f(z) = \sum_{m=-\infty}^{\infty} a_m z^m,$$

where the coefficients a_m are given by

$$a_m = \frac{1}{2\pi i} \int_{S(0,r)} \frac{f(w)}{w^{m+1}} dw$$

with arbitrary $r \in (0, 2)$. For $m = -n-1, n = 0, 1, \ldots$, and $r = 1$, we have

$$a_{-n-1} = \frac{1}{2\pi i} \int_{S(0,1)} f(w) w^n dw = 0, n = 0, 1, 2, \ldots,$$

so that the principal part of the Laurent expansion vanishes. By definition this means that f has a removable singularity at $z = 0$, as required.

(c) Partial fractions:

$$f(z) = \frac{1}{1+2i} \left(\frac{2i}{z-2i} + \frac{1}{z+1} \right).$$

Expand $f(z)$ in powers of $z+1$ under the assumption $0 < |z+1| < \sqrt{5}$. The first term expands as follows:

$$\frac{1}{z-2i} = \frac{1}{z+1-(1+2i)} = -\frac{1}{(1+2i)\left(1 - \dfrac{z+1}{1+2i}\right)}$$

$$= -\frac{1}{1+2i} \sum_{k=0}^{\infty} \frac{(z+1)^k}{(1+2i)^k}.$$

The second term is already in the required form: $(z+1)^{-1}$. Thus,

$$f(z) = -2i \sum_{k=0}^{\infty} \frac{(z+1)^k}{(1+2i)^{k+2}} + \frac{1}{(1+2i)(1+z)}.$$

Let $|z| > 2$. Then

$$\frac{1}{z-2i} = \frac{1}{z\left(1 - \frac{2i}{z}\right)} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{(2i)^k}{z^k},$$

and

$$\frac{1}{z+1} = \frac{1}{z}\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}(-1)^k \frac{1}{z^k}.$$

Thus

$$f(z) = \frac{1}{1+2i} \sum_{k=0}^{\infty} \left((2i)^{k+1} + (-1)^k \right) \frac{1}{z^{k+1}}.$$

2. (a) What does it mean for a function f to be *holomorphic* on the domain $\Omega \subset \mathbb{C}$?
(b) Describe three types of isolated singularities of a function f by explaining how they are related to the principal part of its Laurent expansion.
(c) What type of singularity does the function

$$\frac{\cos z - 1}{z^2}$$

have at the point $z_0 = 0$? Explain your answer.
(d) Suppose that f is holomorphic on a domain Ω and that $|f(z)|$ is constant. Show that f is a constant function.

Solution.

(a) A function f is said to be holomorphic in a domain Ω if it is differentiable at every point of Ω.

(b) If a function f has an isolated singularity at a point z_0, then it has the Laurent expansion of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

If $c_k = 0$ for $k < -M, M > 0$ and $c_{-M} \neq 0$, then the function is said to have a pole of order M.
If there is no such number $N \in \mathbb{Z}$ that $c_k = 0$ for all $k < N$, then the singularity is said to be essential.
If $c_k = 0$ for all $k < 0$, then the singularity is said to be removable.

(c) Expand the function in the Laurent series at $z_0 = 0$:

$$\frac{\cos z - 1}{z^2} = \frac{1}{z^2}\left(1 - \frac{z^2}{2} + \cdots - 1\right) = \frac{1}{2} + \ldots .$$

Thus the singularity is removable.

(d) Suppose that $|f(z)| = c$ for all $z \in \Omega$ with some $c \geq 0$. If $c = 0$, than $f = 0$. Suppose that $c \neq 0$. Since $|f| = c$, we have $u^2 + v^2 = c^2$, where u, v are the real and imaginary parts of the function f, i.e. $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$. Thus, differentiating w.r.t. x, and then w.r.t. y, we get

$$uu_x + vv_x = 0, \quad uu_y + vv_y = 0, \qquad (1)$$

whence, by Cauchy-Riemann equations,

$$uu_x - vu_y = 0, uu_y + vu_x = 0,$$

and hence

$$\begin{cases} u^2 u_x - vuu_y = 0, \\ v^2 u_x + uvu_y = 0, \end{cases}$$

and

$$\begin{cases} uvu_x - v^2 u_y = 0, \\ vuu_x + u^2 u_y = 0, \end{cases}$$

This implies that

$$0 = (u^2 + v^2)u_x = c^2 u_x, \quad 0 = (u^2 + v^2)u_y = c^2 u_y,$$

so that $u_x = 0$, $u_y = 0$ (as $c \neq 0$). By Cauchy-Riemann equations, $v_x = v_y = 0$. Thus $f'(z) = 0$, and therefore $f \equiv$ *constant*.

3. (a) Calculate the integral

$$\int_0^\pi \frac{1}{(\alpha+\cos\theta)^2}d\theta\ ,$$

where $\alpha > 1$.

(b) Suppose that f is analytic at the point $w \in \mathbb{C}$, and define

$$g(z) = \frac{f(z) + wf'(w) - zf'(w) - f(w)}{(z-w)^2}.$$

Show that g has a removable singularity at w.

Solution.

(a) Note that

$$\int_0^\pi \frac{1}{(\alpha+\cos\theta)^2}d\theta = \frac{1}{2}I,\ I := \int_0^{2\pi} \frac{1}{(\alpha+\cos\theta)^2}d\theta.$$

Change the variable of integration: $z = e^{i\theta}$, so that

$$\cos\theta = \frac{1}{2}(z+z^{-1}),\ dz = izd\theta, \theta = \frac{1}{iz}dz.$$

Thus

$$I = \int_{|z|=1} \frac{1}{\left(\alpha + \frac{z+z^{-1}}{2}\right)^2}\frac{1}{iz}dz = \frac{4}{i}\int_{|z|=1}\frac{z}{(z^2+2\alpha z+1)^2}dz.$$

The denominator has roots at

$$z_1 = \sqrt{\alpha^2-1}-\alpha,\ z_2 = -\sqrt{\alpha^2-1}-\alpha.$$

Note that $|z_1| < 1$, and $|z_2| > 1$, so that by Cauchy's Residue Theorem,

$$I = \frac{4}{i}\ 2\pi i\ Res(f, z_1),\ \ f(z) = \frac{z}{(z^2+2\alpha z+1)^2}.$$

Since the pole is of order 2, in order to find the residue, calculate the limit:

$$\lim_{z\to z_1}\frac{d}{dz}(z-z_1)^2 f(z) = \lim_{z\to z_1}\frac{d}{dz}\frac{z}{(z-z_2)^2} = \lim_{z\to z_1}\frac{-z-z_2}{(z-z_2)^3}$$

$$= -\frac{z_1+z_2}{(z_1-z_2)^3} = \frac{2\alpha}{(2\sqrt{\alpha^2-1})^3}.$$

Thus

$$I = \frac{4}{i}\, 2\pi i \frac{2\alpha}{(2\sqrt{\alpha^2-1})^3} = \frac{2\pi\alpha}{(\sqrt{\alpha^2-1})^3},$$

and hence

$$\int_0^\pi \frac{1}{(\alpha+\cos\theta)^2}\,d\theta = \frac{\pi\alpha}{(\sqrt{\alpha^2-1})^3}.$$

Standard

(b) Let us find the Laurent expansion for the function g at the point w. If

$$f(z) = \sum_{m=0}^{\infty} a_m (z-w)^m,$$

then $f(w) = a_0$, $f'(w) = a_1$, so that

$$(z-w)^2 g(z) = \sum_{m=0}^{\infty} a_m (z-w)^m - a_0 - a_1(z-w) = \sum_{m=2}^{\infty} a_m (z-w)^m.$$

This means that

$$g(z) = \sum_{m=0}^{\infty} a_{m+2} (z-w)^m,$$

and therefore, g has a removable singularity at w, as required.

4. Calculate the integral

$$I = \int_{-\infty}^{\infty} \frac{\cos x - 1}{x^2} dx.$$

Solution. The function

$$f(z) = \frac{\cos z - 1}{z^2}$$

is analytic on $\mathbb{C} \setminus \{0\}$ and has a removable singularity at $z = 0$, so the integral does not change its value if we replace the real line by an "indented" path

$$C = (-\infty, -\delta] \cup C_\delta \cup [\delta, \infty),$$

where $C_\delta = \{\delta e^{i(\pi - t)},\ t \in [0, \pi]\}$ with some $\delta > 0$. Thus

$$I = \lim_{R \to \infty} \int_{\Gamma_R^{(0)}} f(z) dz, \quad \Gamma_R^{(0)} = \{z \in C : |\operatorname{Re} z| \le R\}.$$

Represent $f(z)$ as

$$f(z) = g_1(z) + g_2(z),\ g_1(z) = \frac{e^{iz} - 1}{2z^2},\ g_2(z) = \frac{e^{-iz} - 1}{2z^2},$$

and do calculations separately for g_1 and g_2.

To find the integral for g_1, we close the contour in the upper half-plane by introducing the semi-circular path

$$\Gamma_R^{(+)} = \{z = Re^{i\theta},\ \theta \in [0, \pi]\},\ R > \delta.$$

Then the function g_1 is analytic inside the contour

$$\Gamma_R = \Gamma_R^{(0)} \cup \Gamma_R^{(+)},$$

and, by Jordan's Lemma,

$$\int_{\Gamma_R^{(+)}} \frac{e^{iz}}{z^2} dz \to 0, \quad R \to \infty.$$

Since $\max_{|z|=R} |z|^{-2} = R^{-2}$, we also have

$$\int_{\Gamma_R^{(+)}} \frac{1}{z^2} dz \to 0, R \to \infty.$$

Thus $\int_{\Gamma_R} g_1(z)dz = 0$, and hence

$$\int_{\Gamma_R^{(0)}} g_1(z)dz = -\int_{\Gamma_R^{(+)}} g_1(z)dz \to 0, \quad R \to \infty. \tag{2}$$

To find the integral for g_2, we close the contour in the lower half-plane by introducing the semi-circular path

$$\Gamma_R^{(-)} = \{z = Re^{i\theta},\ \theta \in [-\pi, 0]\},\ R > \delta.$$

Then inside the positively oriented contour

$$\tilde{\Gamma}_R = \Gamma_R^{(-)} \cup (-\Gamma_R^{(0)})$$

the function g_2 has a pole at $z = 0$. This is a simple pole. Indeed,

$$g_2(z) = \frac{e^{-iz} - 1}{2z^2} = \frac{-iz - \frac{z^2}{2} + \ldots}{2z^2} = -\frac{i}{2z} - \frac{1}{4} + \ldots.$$

From here one sees that $Res(g_2, 0) = -i/2$, and thus by Cauchy's Residue Theorem,

$$\int_{\tilde{\Gamma}_R} g_2(z)dz = \pi.$$

By the same argument as before,

$$\int_{\Gamma_R^{(-)}} g_2(z)dz \to 0, \quad R \to \infty.$$

Thus

$$\int_{\Gamma_R^{(0)}} g_2(z)dz = -\int_{\tilde{\Gamma}_R} g_2(z)dz + \int_{\Gamma_R^{(-)}} g_2(z)dz \to -\pi, \quad R \to \infty.$$

Together with (2) this implies that

$$\int_{\mathbb{R}} f(x)dx = \int_C (g_1(z) + g_2(z))dz = -\pi.$$

Standard

5. (a) Let f be analytic on a domain Ω, and let $\overline{D}(a,R) \subset \Omega$ with some $a \in \mathbb{C}$.

i. Prove that

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+Re^{it})dt.$$

ii. Evaluate the integral

$$\int_{r<|z-a|<R} f(x+iy)dxdy,$$

where $r < R$.

(b) Suppose that g is analytic on the disk $D(a,R)$, and that $|g(z)| \le |g(a)|$ for all $z \in D(a,R)$. Show that $g(z) = g(a)$, for all $z \in D(a,R)$.

Solution.

(a) i. Use the Cauchy formula:

$$f(a) = \frac{1}{2\pi i}\int_{S(a,R)} \frac{f(z)}{z-a}dz.$$

Rewrite the integral, using the parametrisation $z = a + Re^{it}$, $t \in [0, 2\pi]$:

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+Re^{it})}{Re^{it}} iRe^{it}dt = \frac{1}{2\pi}\int_0^{2\pi} f(a+Re^{it})dt, \qquad (3)$$

as required.

ii. Use polar coordinates and formula (3):

$$\int_{r<|z-a|<R} f(x+iy)dxdy = \int_{r<\rho<R}\int_0^{2\pi} f(a+\rho e^{i\phi})d\phi \rho d\rho$$

$$= \int_r^R 2\pi f(a)\rho d\rho = \pi f(a)(R^2 - r^2).$$

(b) Let $0 < r < R$. By Part (a) of the question,

$$g(a) = \frac{1}{2\pi}\int_0^{2\pi} g(a + re^{i\theta})d\theta.$$

Thus

$$|g(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} |g(a + re^{i\theta})|d\theta. \tag{4}$$

Suppose that for some θ_0 the value $|g(a + re^{i\theta_0})|$ is strictly less than $|g(a)|$. Then by continuity of g there is an interval around θ_0, say $(\theta_0 - \delta, \theta_0 + \delta)$ where $|g(a + re^{i\theta})| < |g(a)|$. Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} |g(a + re^{i\theta})|d\theta < |g(a)|.$$

Together with (4) this leads to a contradiction, and hence $|g(z)|$ is constant on the disk. It is known that an analytic function with a constant modulus is constant itself, so that $g(z) = g(a)$ as required.

6. (a) Let h be a function holomorphic on the punctured disk $D'(a, R)$. Suppose that $|h(z)| \leq M$ for all $z \in D'(a, R)$ with some positive constant M. Show that

   i. h has a removable singularity at a,

   ii. after removing the singularity, the obtained holomorphic function h satisfies the bound $|h(z)| \leq M$ for all $z \in D(a, R)$.

   (b) Suppose that f and g are entire functions, and that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a number $\alpha \in \mathbb{C}$ such that $f(z) = \alpha g(z)$ for all $z \in \mathbb{C}$.

Solution.

(a) i. We need to show that in the Laurent expansion

$$h(z) = \sum_{m=-\infty}^{\infty} a_m (z-a)^m,$$

the coefficients a_m with $m \leq -1$, vanish. By Laurent Theorem, for any $m = 1, 2, \dots$, we have

$$a_{-n} = \frac{1}{2\pi i} \int_{S(0,r)} h(z)(z-a)^{n-1} dz$$

for arbitrary $r \in (0, R)$. Since $|h(z)| \leq M$, we can estimate:

$$|a_{-n}| \leq \frac{1}{2\pi} M r^{n-1} 2\pi r = M r^n.$$

Since $r > 0$ is arbitrary and $n \geq 1$, it follows that $a_{-n} = 0$, as required.

ii. After defining $h(a) = a_0$, the function h becomes holomorphic on $D(a, R)$. Thus by the Maximum Modulus Principle,

$$\max_{z \in \overline{D}(a,r)} |h(z)| = \max_{|z|=r} |h(z)|$$

for any $r \in (0, R)$. The right-hand-side of the above identity does not exceed M by hypothesis, an hence $|h(z)| \leq M$ for all $z \in D(a, R)$ as required.

Standard

(b) Let $h(z) = f(z)/g(z)$. This function is holomorphic outside the roots of g, and $|h(z)| \leq 1$. By Part (a), the roots of g are removable singularities of h, and after removing the singularities, the function h becomes entire, and it satisfies the bound $|h(z)| \leq 1$ for all $z \in \mathbb{C}$. By Liouville's Theorem, $h(z) \equiv \alpha$ with some constant $\alpha \in \mathbb{C}$, so that $f(z) = \alpha g(z)$ as required.